

# The Penalty Cell-Centered Finite Element Scheme For Stokes Problem On General Meshes

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## Abstract

The paper is devoted to the penalty cell-centered finite element scheme (pFECC) on general meshes for the stationary Stokes problems with an incompressible variable viscosity and Dirichlet boundary conditions. In the objectives of this work, we show the rigorous mathematical analysis including the existence, the uniqueness of a discrete solution of the problem, the symmetric and the positive definite stiffness matrix, convergence of the pFECC scheme.

*Keywords:* The cell-centered finite element scheme; Penalty method; General grids; Volumetric locking; The stationary Stokes equations for an incompressible variable viscous fluid.

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## 1. Introduction

Let  $\Omega$  be a open, bounded domain of  $\mathbb{R}^2$  with the boundary  $\partial\Omega$ . We consider the stationary Stokes problem for an incompressible variable viscosity in  $\Omega$ : find an approximation weak solution of  $\mathbf{u} = (u^{(1)}, u^{(2)}) \in (H_0^1(\Omega))^2$  and  $p \in L^2(\Omega)$ , to the following problem

$$\begin{aligned} -\operatorname{div}(2\mu(\mathbf{x})\boldsymbol{\mathcal{E}}(\mathbf{u})) + \nabla p &= \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div}(\mathbf{u}) &= 0, & \text{in } \Omega, \\ \mathbf{u} &= 0, & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where the velocity  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$ , has the two components  $u^{(1)}$ ,  $u^{(2)}$ , the pressure  $p$  defined over  $\Omega$ , the strain tensor related to the displacement is defined by  $\boldsymbol{\mathcal{E}}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$  and  $\mathbf{f}$  are the body forces per unit mass.

In the physical model (1), the variable viscosity (the measure of a fluid's ability to

resist gradual deformation by shear or tensile stresses) of non-Newtonian fluids is dependent on shear rate or shear rate history. This interesting physical model is appeared in many commonly found substances such as ketchup, custard, toothpaste, starch suspensions, paint, blood, and shampoo...In this paper, we also assume that the viscosity  $\mu : \Omega \rightarrow \mathbb{R}$  in (1) is piecewise Lipschitz-continuous on the domain  $\overline{\Omega}$  and there exists  $\lambda, \underline{\lambda}, \overline{\lambda}$  such that

$$\underline{\lambda} \leq \mu(\mathbf{x}) \leq \overline{\lambda}, \quad \text{for a.e } \mathbf{x} \in \Omega \quad (2)$$

and

$$|\mu(\mathbf{x}) - \mu(\overline{\mathbf{x}})| \leq \lambda |\mathbf{x} - \overline{\mathbf{x}}|, \quad \text{for all } \mathbf{x}, \overline{\mathbf{x}} \in \overline{\Omega} \quad (3)$$

With the important physical role of the Stokes equations (2), there are many numerical schemes have been extensively studied: see [11, 20, 21, 22, 13, 12] and references therein. Among different schemes, finite element schemes and finite volume schemes are frequently used for mathematical or engineering studies.

## 2. Stokes problems

Under hypotheses  $\mathbf{f} = (f^{(1)}, f^{(2)}) \in (L^2(\Omega))^2$ , (2) and (3), let

$$\mathbb{H}(\Omega) = \left\{ \mathbf{v} = (v^{(1)}, v^{(2)}) \in (H_0^1(\Omega))^2, \operatorname{div}(\mathbf{v}) = \frac{\partial v^{(1)}}{\partial x_1} + \frac{\partial v^{(2)}}{\partial x_2} = 0 \right\}, \quad (4)$$

then the weak solution  $\mathbf{u} = (u^{(1)}, u^{(2)})$  of (1) (see e.g. [1, 4]) must be satisfied

$$\begin{cases} \mathbf{u} = (u^{(1)}, u^{(2)}) \in \mathbb{H}(\Omega), \\ \int_{\Omega} \eta(\mathbf{x}) \nabla \mathbf{u}(\mathbf{x}) : \nabla \mathbf{v}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \mathbf{f}(\mathbf{x}) \mathbf{v}(\mathbf{x}) d\mathbf{x}, \quad \forall \mathbf{v} = (v^{(1)}, v^{(2)}) \in \mathbb{H}(\Omega) \end{cases} \quad (5)$$

with  $\mathbf{x} = (x_1, x_2)$  and  $\nabla \mathbf{u}(\mathbf{x}) : \nabla \mathbf{v}(\mathbf{x}) = \nabla u^{(1)}(\mathbf{x}) \cdot \nabla v^{(1)}(\mathbf{x}) + \nabla u^{(2)}(\mathbf{x}) \cdot \nabla v^{(2)}(\mathbf{x})$ .

In order to study convergence of the approximate solution, we need the regularity of the weak solution  $(\mathbf{u}, p)$ . Thank to Lemma 5.2.5 of [1], the author proved that if the viscosity  $\mu$  belongs to  $C^2(\Omega)$ , then the solution  $(\mathbf{u}, p)$  satisfy

$$\mathbf{u} \in (H^2(\Omega))^2 \text{ and } p \in H^1(\Omega). \quad (6)$$

## 3. The penalty cell-centered finite element framework

The cell-centered finite element scheme (FECC), which was firstly introduced by Christophe and Ong [12], was applied into the diffusion problems on general meshes. To develop the idea of the scheme for Stokes problems, we combine the FECC and the stabilization inspired by the well-known penalty method [7, 11] in the finite element framework.

### 3.1. Discretization of the domain $\Omega$

For a given  $\Omega$  be an open bounded polygonal set of  $\mathbb{R}^2$  with boundary  $\partial\Omega$ . In order to partition the domain  $\Omega$ , we use the three families  $\mathcal{D}(\mathcal{M}, \mathcal{E}, \mathcal{P})$ ,  $\mathcal{D}^*(\mathcal{M}^*, \mathcal{E}^*, \mathcal{P}^*)$  and  $\mathcal{D}^{**}(\mathcal{M}^{**}, \mathcal{E}^{**}, \mathcal{P}^{**})$  constructed in [12]. The first mesh  $\mathcal{M}$  is assumed that any line which connects two mesh points of two adjacent elements of  $\mathcal{M}^*$  intersects with the common edge of these elements at the unique point. Without loss of generality, we can choose each dual mesh point of  $\mathcal{M}^*$  located at a vertex of  $\mathcal{M}$ . We introduce elements of  $\mathcal{M}^{**}$  denoted by  $L^{**}$  or  $K^{**}$  or  $T_{K^*, \sigma}$ , where let an edge  $\sigma$  of  $K^* \in \mathcal{M}^*$ , a triangle  $T_{K^*, \sigma}$  stays in  $K^*$ .

#### 3.1.1. Geometrical Conditions For Regular Meshes

To study the convergence of the scheme, we will need the size of the third mesh  $\mathcal{M}^{**}$  defined by

$$h = \sup_{K^{**} \in \mathcal{M}^{**}} \text{diam}(K^{**}), \quad (7)$$

where  $\text{diam}(K^*)$  indicates a diameter of the circumscribed circle of  $K^*$ .

And the regularity of the three meshes is required by the existence of positive real numbers  $C_i$  such that

$$\text{card}(\mathcal{E}_{K^*}) \leq C_1, \quad \text{for all } K^* \in \mathcal{M}^*, \quad (8)$$

where  $\mathcal{E}_{K^*}$  is a set of all edges of  $K^*$ . This corresponds to the condition

$$\text{card}(\mathcal{M}_{K^*}^{**}) \leq C_1 \quad \text{for all } K^* \in \mathcal{M}^*$$

in which  $\mathcal{M}_{K^*}^{**}$  is a set of all element of  $\mathcal{M}^{**}$  staying in  $K^*$ .

$$\text{diam}^2(K^*) \leq C_2 m(K^*), \quad \forall K^* \in \mathcal{M}^*. \quad (9)$$

$$\text{diam}^2(T) \leq C_3 m(L^{**}), \quad \forall L^{**} \in \mathcal{M}^{**}. \quad (10)$$

Besides, we also have another condition for the third mesh  $\mathcal{M}^{**}$ :

*Inverse assumption: There exists constant  $\zeta_{\mathcal{D}^{**}} > 0$  such that*

$$\max_{K^{**} \in \mathcal{M}^{**}} \frac{h}{\text{diam}(K^{**})} \leq \zeta_{\mathcal{D}^{**}} \quad \text{for all } h > 0. \quad (11)$$

### 3.2. Unknowns and Discrete operators

We will express the new scheme in the weak form; to this aim, let us firstly define the sets containing the discrete unknowns, the discrete operators, the discrete gradient, and the discrete divergence:

For given two neighboring elements of the first mesh  $\mathcal{M}$ , we assume that the line joining their primary mesh points can be intersected their common edge. With this assumption, let  $\sigma \in \mathcal{E}_{\text{int}}$  such that  $\mathcal{M}_\sigma = \{K, L\}$ , the three points  $\mathbf{x}_K, \mathbf{x}_L \in \mathcal{P}$  and  $\mathbf{x}_{K^*} \in \mathcal{P}^*$  can generate a triangular element  $(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})$  of  $\mathcal{M}^{**}$ . On this triangle, we take the unknown values  $\mathbf{u}_{K^*} = (u_{K^*}^{(1)}, u_{K^*}^{(2)})$ ,  $\mathbf{u}_K = (u_K^{(1)}, u_K^{(2)})$ ,  $\mathbf{u}_L = (u_L^{(1)}, u_L^{(2)})$  of the velocity  $\mathbf{u} = (u^{(1)}, u^{(2)})$  at  $\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_L$ . Besides, we have a notation  $\mathbf{u}_\sigma^{K^*}$  (a temporary unknown) seen as a value of  $\mathbf{u}$  at  $\mathbf{x}_\sigma$ , where the point  $\mathbf{x}_\sigma$  is an intersecting point between the line joining two mesh points  $\mathbf{x}_K, \mathbf{x}_L$  and the internal edge  $\sigma$ .

From these values, we introduce the following discrete velocity space

**Definition 3.1:** *Let us define the discrete function space  $\mathcal{H}_\mathcal{D}$  as the set of all  $((\mathbf{u}_K)_{K \in \mathcal{M}}, (\mathbf{u}_{K^*})_{K^* \in \mathcal{M}^*})$ ,  $\mathbf{u}_K \in \mathbb{R}^2$  for all  $K \in \mathcal{M}$  and  $\mathbf{u}_{K^*} \in \mathbb{R}^2$  for all  $K^* \in \mathcal{M}^*$ . Moreover, the value  $\mathbf{u}_{K^*}$  is equal to 0, while a mesh point  $\mathbf{x}_{K^*}$  belongs to the boundary  $\partial\Omega$ .*

and the discrete pressure space  $\mathcal{L}_\mathcal{D}$ :

**Definition 3.2:** *The space  $\mathcal{L}_\mathcal{D}$  contains all piecewise constant functions on the dual mesh  $\mathcal{M}^*$ .*

$$\mathcal{L}_\mathcal{D} = \left\{ q : \Omega \rightarrow \mathbb{R} \mid q(\mathbf{x}) = \sum_{K^* \in \mathcal{M}^*} q_{K^*} \chi_{K^*}(\mathbf{x}) \right\},$$

with the characteristic function  $\chi_{K^*}$ , for each  $K^* \in \mathcal{M}^*$ .

From the definition of the two discrete spaces  $\mathcal{H}_\mathcal{D}$  and  $\mathcal{L}_\mathcal{D}$ , we construct a discrete gradient  $\nabla_{\mathcal{D}, \Lambda} u_h^{(i)}$  and the interpolation operator  $P_{(K^*, K, L)}(u_h^{(i)})$ ,  $i = 1, 2$ , on two sub-triangles of the triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_L)$ , where their definitions are taken into account the variable viscosity  $\mu(\mathbf{x})$  and an element  $\mathbf{u}_h = ((\mathbf{u}_K)_{K \in \mathcal{M}}, (\mathbf{u}_{K^*})_{K^* \in \mathcal{M}^*}) \in \mathcal{H}_\mathcal{D}$ , as follows:

The interpolation operator for each its element  $u_h^{(i)}$ ,  $i = 1, 2$ , is defined by

$$P_{(K^*, K, L)}(u_h^{(i)}) : (\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_L) \rightarrow \mathbb{R},$$

such that it is continuous, piecewise linear on  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)$  and  $(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)$  (two sub-triangles of  $(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})$ ).

- on the sub-triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)$

$$P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}) = \begin{cases} u_K^{(i)} & \mathbf{x} = \mathbf{x}_K, \\ u_{K^*}^{(i)} & \mathbf{x} = \mathbf{x}_{K^*}, \\ u_{\sigma, K^*}^{(i)} & \mathbf{x} = \mathbf{x}_\sigma. \end{cases}$$

$$\begin{aligned} \nabla_{\mathcal{D}, \mu} u^{(i)} &= \nabla_{\mathcal{D}, \mu} P_{(K^*, K, L)}(u_h^{(i)}) \\ &= \frac{-P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}_\sigma) n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]} - P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}_K) n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^K - P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}_{K^*}) n_{[\mathbf{x}_\sigma, \mathbf{x}_K]}}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)}} \\ &= \frac{-u_{\sigma, K^*}^{(i)} n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]} - u_K^{(i)} n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^K - u_{K^*}^{(i)} n_{[\mathbf{x}_\sigma, \mathbf{x}_K]}}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)}}, \end{aligned}$$

where  $n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^K$  is outer normal vector to the triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)$ . The length of vector  $n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^K$  is equal to the length of segment  $[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]$ . If  $\mathbf{x}_\sigma$  belongs to boundary  $\partial\Omega$  then  $u_{\sigma, K^*}^{(i)} = 0$ . A notation  $m_{(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)}$  is the area of a triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)$

- on the sub-triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)$

$$P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}) = \begin{cases} u_L^{(i)} & \mathbf{x} = \mathbf{x}_L, \\ u_{K^*}^{(i)} & \mathbf{x} = \mathbf{x}_{K^*}, \\ u_{\sigma, K^*}^{(i)} & \mathbf{x} = \mathbf{x}_\sigma. \end{cases}$$

$$\begin{aligned} \nabla_{\mathcal{D}, \mu} u_h^{(i)} &= \nabla_{\mathcal{D}, \mu} P_{(K^*, K, L)}(u_h^{(i)}) \\ &= \frac{-P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}_\sigma) n_{[\mathbf{x}_{K^*}, \mathbf{x}_L]} - P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}_L) n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^L - P_{(K^*, K, L)}(u_h^{(i)})(\mathbf{x}_{K^*}) n_{[\mathbf{x}_\sigma, \mathbf{x}_L]}}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)}} \\ &= \frac{-u_{\sigma, K^*}^{(i)} n_{[\mathbf{x}_{K^*}, \mathbf{x}_L]} - u_L^{(i)} n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^L - u_{K^*}^{(i)} n_{[\mathbf{x}_\sigma, \mathbf{x}_L]}}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)}}, \end{aligned}$$

where  $n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^L$  is outer normal vector to the triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)$ . The length of vector  $n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]}^L$  equal to the length of segment  $[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]$ . A notation  $m_{(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)}$  is the area of a triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)$ .

These definitions depend on  $u_{\sigma,K^*}^{(i)}$ , but this temporary unknown can be fixed by imposing the Local Conservativity of the Fluxes condition, i.e

$$\mu_K \left( \nabla_{\mathcal{D},\mu} u_h^{(i)} \right)_{|(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)} \cdot n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^K} + \mu_L \left( \nabla_{\mathcal{D},\mu} u_h^{(i)} \right)_{|(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)} \cdot n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^L} = 0, \quad (12)$$

where  $\mu_K, \mu_L$  are the average values of  $\mu$  on  $K$  and  $L$ .

Equation (12) leads to the following linear combination depended on  $\{u_K^{(i)}, u_L^{(i)}, u_{K^*}^{(i)}\}$

$$u_{\sigma,K^*}^{(i)} = \beta_K^{K^*,\sigma} u_K^{(i)} + \beta_L^{K^*,\sigma} u_L^{(i)} + \beta_{K^*}^{K^*,\sigma} u_{K^*}^{(i)}, \quad \text{for each } i = 1, 2, \quad (13)$$

where the coefficients are written by

$$\begin{aligned} \beta_K^{K^*,\sigma} &= \left( \frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^K}^T \mu_K n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^K})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)}} \right) / \left( -\frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^K}^T \mu_K n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)}} - \frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^L}^T \mu_L n_{[\mathbf{x}_{K^*}, \mathbf{x}_L]})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)}} \right), \\ \beta_L^{K^*,\sigma} &= \left( \frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^L}^T \mu_L n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^L})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)}} \right) / \left( -\frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^K}^T \mu_K n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)}} - \frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^L}^T \mu_L n_{[\mathbf{x}_{K^*}, \mathbf{x}_L]})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)}} \right), \\ \beta_{K^*}^{K^*,\sigma} &= 1 - \beta_K^{K^*,\sigma} - \beta_L^{K^*,\sigma}. \end{aligned}$$

From Equation (13), the unknown  $u_{\sigma,K^*}^{(i)}$  is computed by  $u_K^{(i)}$ ,  $u_{K^*}^{(i)}$  and  $u_L^{(i)}$ . Thus, the discrete gradient  $\nabla_{\mathcal{D},\mu} u_h^{(i)}$  on  $(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})$  only depends on these three values.

**Hypothesis 3.1:** we assume

$$\left( -\frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^K}^T \mu_K n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_\sigma)}} - \frac{(n_{[\mathbf{x}_\sigma, \mathbf{x}_{K^*}]^L}^T \mu_L n_{[\mathbf{x}_{K^*}, \mathbf{x}_L]})}{2m_{(\mathbf{x}_{K^*}, \mathbf{x}_L, \mathbf{x}_\sigma)}} \right) \neq 0. \quad (14)$$

Note that if the mesh points  $\mathbf{x}_K$  or  $\mathbf{x}_L$  are moved slightly, the value of the right hand side in (14) is changed. This asserts that the hypothesis 3.1 is easy to be satisfied.

Using the definition of the discrete gradient  $\nabla_{\mathcal{D}} u_h^{(i)}$ , for each element of  $\mathbf{u}_h$ , we define the discrete divergence on the triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_L)$  by

$$\text{div}_{\mathcal{D},\mu}(\mathbf{u}_h) = \nabla_{\mathcal{D},\mu} u_h^{(1)} \cdot \mathbf{e}_1 + \nabla_{\mathcal{D},\mu} u_h^{(2)} \cdot \mathbf{e}_2 \quad (15)$$

with  $\mathbf{e}_i$ ,  $i = 1, 2$ , the basis unit vector corresponding to the  $i$ -th coordinate.

### 3.3. Discrete variational formulation

The existence and uniqueness of weak solution of the Stokes problem (1) was stated in the section 2. However, in order to apply the pFECC scheme, we would like to implement another usual variational formulation for the problem (1), as follows: Find the velocity  $\mathbf{u} \in (H_0^1(\Omega))^2$  and the pressure  $p \in L_0^2(\Omega)$  such that

$$\begin{cases} \int_{\Omega} \mu(\mathbf{x}) \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} - \int_{\Omega} \operatorname{div}(\mathbf{v}) p \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, & \mathbf{v} \in (H_0^1(\Omega))^2 \\ \int_{\Omega} \operatorname{div}(\mathbf{u}) q \, d\mathbf{x} = 0, & q \in L_0^2(\Omega), \end{cases} \quad (16)$$

with the space

$$L_0^2(\Omega) = \{p \in L^2(\Omega) \mid \int_{\Omega} p \, d\mathbf{x} = 0\}.$$

Applying the pFECC scheme into the velocity-pressure system (16), we will look for the discrete velocity  $\mathbf{u}_h \in \mathcal{H}_{\mathcal{D}}$ , the discrete pressure  $p_h \in \mathcal{L}_{\mathcal{D}}$  satisfying the following problem

$$\begin{aligned} \int_{\Omega} \mu(\mathbf{x}) \nabla_{\mathcal{D},\mu} \mathbf{u}_h : \nabla_{\mathcal{D},\mu} \mathbf{v}_h \, d\mathbf{x} - \int_{\Omega} \operatorname{div}_{\mathcal{D},\mu}(\mathbf{v}_h) p_h \, d\mathbf{x} &= \int_{\Omega} \mathbf{f} \cdot P(\mathbf{v}_h) \, d\mathbf{x}, \\ \text{with } P(\mathbf{v}_h) &= (P(v_h^{(1)}), P(v_h^{(2)})), \text{ for all } \mathbf{v}_h = (v^{(1)}, v^{(2)}) \in \mathcal{H}_{\mathcal{D}}, \end{aligned} \quad (17)$$

and

$$\int_{\Omega} \operatorname{div}_{\mathcal{D},\mu}(\mathbf{u}_h) q_h \, d\mathbf{x} = -\lambda h \int_{\Omega} p_h q_h \, d\mathbf{x}, \quad \text{for all } q_h \in \mathcal{L}_{\mathcal{D}}, \quad (18)$$

where  $\nabla_{\mathcal{D},\mu} \mathbf{u}_h : \nabla_{\mathcal{D},\mu} \mathbf{v}_h$  is defined by

$$\nabla_{\mathcal{D},\mu} \mathbf{u}_h : \nabla_{\mathcal{D},\mu} \mathbf{v}_h = \nabla_{\mathcal{D},\mu} u_h^{(1)} \cdot \nabla_{\mathcal{D},\mu} v_h^{(1)} + \nabla_{\mathcal{D},\mu} u_h^{(2)} \cdot \nabla_{\mathcal{D},\mu} v_h^{(2)}. \quad (19)$$

$$\operatorname{div}_{\mathcal{D},\mu} \mathbf{u}_h = \partial_{\mathcal{D},\mu}^{(1)} u_h^{(1)} + \partial_{\mathcal{D},\mu}^{(2)} u_h^{(2)} \quad (20)$$

where the discrete partial divergence  $\partial_{\mathcal{D},\mu}^j u_h^{(i)}$  corresponds to a discretization of the partial divergence  $\frac{\partial u_h^{(i)}}{\partial x_j}$ , defined by

$$\partial_{\mathcal{D},\mu}^j u_h^{(i)} = \nabla_{\mathcal{D},\mu} u_h^{(i)} \cdot \mathbf{e}_j \quad \text{for } i, j \in \{1, 2\}. \quad (21)$$

### 3.4. The linear algebraic systems

Let us describe the three implementation steps to construct the system of linear equations depended on  $\{\mathbf{u}_K\}_{K \in \mathcal{M}}$  and  $\{p_{K^*}\}_{K^* \in \mathcal{M}^*}$ , as follows:

*In the first step:* For each element  $K^* \in \mathcal{M}^*$ , a discrete test pressure function  $q_h \in \mathcal{L}_{\mathcal{D}}$  is only equal to 1 on  $K^*$  and 0 on  $L^* \in \mathcal{M}^*/\{K^*\}$ , Equation(18) is stated by

$$\int_{K^*} \left( \nabla_{\mathcal{D}, \mu} u_h^{(1)} \cdot \mathbf{e}_1 + \nabla_{\mathcal{D}, \mu} u_h^{(2)} \cdot \mathbf{e}_2 \right) d\mathbf{x} = -\lambda h m(K^*) p_{K^*} \quad (22)$$

Besides, Equation (17) is computed with each value of a discrete test velocity function:

- $\mathbf{v}_h = (\{\mathbf{v}_L\}_{L \in \mathcal{M}}, \{\mathbf{v}_{L^*}\}_{L^* \in \mathcal{M}^*})$  satisfies  $\mathbf{v}_{K^*} = (1, 0)$ ;  $\mathbf{v}_M = (0, 0)$  for all  $M \in \{\mathcal{M} \cup \mathcal{M}^*\}/\{K^*\}$ , then this equation is rewritten as

$$\int_{K^*} \left( \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(1)} \right) \cdot \nabla_{\mathcal{D}, \mu} v_h^{(1)} d\mathbf{x} - \int_{K^*} \left( \nabla_{\mathcal{D}, \mu} v_h^{(1)} \cdot \mathbf{e}_1 \right) p d\mathbf{x} = \int_{K^*} f_1 d\mathbf{x}, \quad (23)$$

- $\mathbf{w}_h = (\{\mathbf{w}_L\}_{L \in \mathcal{M}}, \{\mathbf{w}_{L^*}\}_{L^* \in \mathcal{M}^*})$  satisfies  $\mathbf{w}_{K^*} = (0, 1)$ ;  $\mathbf{w}_M = (0, 0)$  for all  $M \in \{\mathcal{M} \cup \mathcal{M}^*\}/\{K^*\}$ , then it is equal to

$$\int_{K^*} \left( \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(2)} \right) \cdot \nabla_{\mathcal{D}, \mu} v_h^{(2)} d\mathbf{x} - \int_{K^*} \left( \nabla_{\mathcal{D}, \mu} v_h^{(2)} \cdot \mathbf{e}_2 \right) p d\mathbf{x} = \int_{K^*} f_2 d\mathbf{x}, \quad (24)$$

We see that Equations (22), (23) and (24) can be represented as three linear combinations only depending on  $\{\mathbf{u}_K\}_{K \in \mathcal{M}}$ ,  $p_{K^*}$ ,  $\mathbf{u}_{K^*} = (u_{K^*}^{(1)}, u_{K^*}^{(2)})$  and  $\mathbf{f}$ . These results help us compute the unknown  $u_{K^*}^{(i)}$  by a linear combination  $\Pi_{K^*}^{(i)}(\{u_K^{(i)}\}_{K \in \mathcal{M}}, \mathbf{f}) + \Psi^{(i)}(p_{K^*})$  with  $i = 1, 2$ .

**Remark 3.1:** The coefficients of  $u_K^{(i)}$  for all  $K \in \mathcal{M}$ ,  $i = 1, 2$ , in the operator  $\Pi_{K^*}^{(i)}$  are the same as those of the function  $\Pi_{K^*}$ , in the second step of [12].

*In the second step:* The unknowns  $\mathbf{u}_{K^*}$ , for all  $K^* \in \mathcal{M}^*$ , in the discrete gradient  $\nabla_{\mathcal{D}, \mu} \mathbf{u}_h$  and the discrete divergence  $\text{div}_{\mathcal{D}, \mu} \mathbf{u}_h$ , are transformed into  $\Pi_{\mathbf{u}_{K^*}}^{(i)}(\{u_K^{(i)}\}_{K \in \mathcal{M}}, \mathbf{f}) + \Psi^{(i)}(p_{K^*})$  and Ep.(22) with  $i = 1, 2$ .

*In the last step:* For each element  $K \in \mathcal{M}$ , a test velocity function in Equation (17) is taken into each following value:



- $\mathbf{v} = (\{\mathbf{v}_K\}_{K \in \mathcal{M}}, \{\mathbf{v}_{K^*}\}_{K^* \in \mathcal{M}^*})$  has  $\mathbf{v}_K = (1, 0)$ ;  $\mathbf{v}_M = (0, 0)$  for all  $M \in \{\mathcal{M} \cup \mathcal{M}^*\} \setminus \{K\}$ , the equation is computed by

$$\int_{\Omega} \left( \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(1)} \right) \cdot \nabla_{\mathcal{D}, \mu} v_h^{(1)} d\mathbf{x} - \int_{\Omega} (\nabla_{\mathcal{D}, \mu} v_h^{(1)} \cdot \mathbf{e}_1) p_h d\mathbf{x} = \int_{\Omega} \mathbf{f}(x) \cdot \mathbf{v}_h d\mathbf{x}, \quad (25)$$

- $\mathbf{w} = (\{\mathbf{w}_K\}_{K \in \mathcal{M}}, \{\mathbf{w}_{K^*}\}_{K^* \in \mathcal{M}^*})$  has  $\mathbf{w}_K = (0, 1)$  and  $\mathbf{w}_M = (0, 0)$  for all  $M \in \{\mathcal{M} \cup \mathcal{M}^*\} \setminus \{K\}$ , then the equation is equal to

$$\int_{\Omega} \left( \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(2)} \right) \cdot \nabla_{\mathcal{D}, \mu} w_h^{(2)} d\mathbf{x} - \int_{\Omega} (\nabla_{\mathcal{D}, \mu} w_h^{(2)} \cdot \mathbf{e}_2) p_h d\mathbf{x} = \int_{\Omega} \mathbf{f}(x) \cdot \mathbf{w}_h d\mathbf{x}, \quad (26)$$

**Remark 3.2:** After the second step, the discrete gradient  $\nabla_{\mathcal{D}, \mu} \mathbf{u}_h$  and divergence  $\text{div}_{\mathcal{D}, \mu} \mathbf{u}_h$  are independent on the unknowns  $\{\mathbf{u}_{K^*}\}_{K^* \in \mathcal{M}^*}$ . Therefore, in two equations (25) and (26), there are not the unknowns  $\{\mathbf{u}_{K^*}\}_{K^* \in \mathcal{M}^*}$ . Additionally, for each  $i = 1, 2$ , we have  $\text{supp}\{\nabla_{\mathcal{D}, \mu}(u_h^i)\}$  belonging to  $\bigcup_{K^* \in \mathcal{M}_K^*} K^*$  with

$$\mathcal{M}_K^* = \{K^* \in \mathcal{M}^* \mid K \cap K^* \neq \emptyset\},$$

which indicates that the stiffness matrix  $\mathbf{A}$  in (27) is sparse

From the above three steps, the stiffness matrix associated to the pFECC scheme for the Stokes problem is generated by

$$\underbrace{\begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C} & (-\lambda h m(K^*) \mathbf{Id}) \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \{\mathbf{u}_K\}_{K \in \mathcal{M}} \\ \{p_{K^*}\}_{K^* \in \mathcal{M}^*} \end{pmatrix} = \mathbf{F} \quad (27)$$

Thanks to Remark 3.1, the matrix  $\mathbf{B}$  are positive definite and symmetric, which is proven in Lemma 3.2 of [12]. Hence, the matrix  $\mathbf{A}$  has the inverse matrix, there then exists the unique solution of the system (27).

#### 4. Consistency and stability of the pFECC method

In this section, we will study the consistency properties of the discrete gradient and the discrete divergence. These results will be necessary to prove the convergence. Let us firstly consider to the divergence operator.

**Lemma 4.1 (Consistency of the discrete divergence)**

Under geometrical conditions for meshes 3.1.1 and assumptions of Lemma 5.1 in [12], then, there exists the positive constant  $C_5$ , such that, for all  $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ , and for each  $K^* \in \mathcal{M}^*$ ,

$$\left| \int_{K^*} \partial_{\mathcal{D},\mu}^{(i)} \bar{u}_h(\mathbf{x}) d\mathbf{x} - \int_{\partial K^*} \bar{u}(x) \mathbf{e}_i \cdot \mathbf{n}(\mathbf{x}) d\mathbf{x} \right| \leq C_4 h^2 [|\bar{u}|_{H^2(K^*)} + \|\nabla \bar{u}\|_{L^2(K^*)}] \quad (28)$$

with  $\bar{u}_h = (\{\bar{u}(\mathbf{x}_K)\}_{K \in \mathcal{M}}, \{\bar{u}(\mathbf{x}_{K^*})\}_{K^* \in \mathcal{M}^*})$  and  $\partial_{\mathcal{D},\mu}^{(i)} \bar{u}_h = \nabla_{\mathcal{D},\mu} \bar{u}_h \cdot \mathbf{e}_i$ .

*Proof.*

For any  $K^* \in \mathcal{M}^*$ , we have

$$\int_{K^*} \partial_{\mathcal{D},\mu}^{(i)} \bar{u}_h d\mathbf{x} = \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} \int_{L^{**}} \partial_{\mathcal{D},\mu}^{(i)} \bar{u}_h d\mathbf{x} \quad (29)$$

with  $\mathcal{M}_{K^*}^{**} = \{L^{**} \in \mathcal{M}^{**} \mid L^{**} \subset K^*\}$ .

Let us consider on any element  $K^{**} \in \mathcal{M}_{K^*}^{**}$ , it is seen as a triangle  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_L)$  having three vertices  $\mathbf{x}_K, \mathbf{x}_L$  with  $K, L \in \mathcal{M}$  and  $\mathbf{x}_{K^*}$  with  $K^* \in \mathcal{M}^*$ .

On this triangle, we compute

$$\int_{(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})} \partial_{\mathcal{D},\mu}^{(i)} \bar{u}_h(\mathbf{x}) d\mathbf{x} = \int_{(\mathbf{x}_K, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \nabla_{\mathcal{D},\mu} \bar{u}_h \cdot \mathbf{e}_i d\mathbf{x} + \int_{(\mathbf{x}_L, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \nabla_{\mathcal{D},\mu} \bar{u}_h \cdot \mathbf{e}_i d\mathbf{x} \quad (30)$$

with

$$\int_{(\mathbf{x}_K, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \nabla_{\mathcal{D},\mu} \bar{u}_h \cdot \mathbf{e}_i d\mathbf{x} = -\frac{1}{2} \left\{ \begin{array}{l} \left( n_{[\mathbf{x}_K, \mathbf{x}_\sigma]} + \beta_{K^*}^{K^*, \sigma} n_{[\mathbf{x}_K, \mathbf{x}_{K^*}]} \right) \cdot \mathbf{e}_i \bar{u}(\mathbf{x}_{K^*}) \\ + \beta_L^{K^*, \sigma} n_{[\mathbf{x}_K, \mathbf{x}_{K^*}]} \cdot \mathbf{e}_i \bar{u}(\mathbf{x}_L) \\ + \left( n_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]}^{(K, K^*, \sigma)} + \beta_K^{K^*, \sigma} n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]} \right) \cdot \mathbf{e}_i \bar{u}(\mathbf{x}_K) \end{array} \right\} \quad (31)$$

and

$$\int_{(\mathbf{x}_L, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \nabla_{\mathcal{D}, \mu} \bar{u} \cdot \mathbf{e}_i d\mathbf{x} = -\frac{1}{2} \left\{ \begin{aligned} & \left( n_{[\mathbf{x}_L, \mathbf{x}_\sigma]} + \beta_{K^*}^{K^*, \sigma} n_{[\mathbf{x}_L, \mathbf{x}_{K^*}]} \right) \cdot \mathbf{e}_i \bar{u}(\mathbf{x}_{K^*}) \\ & + \beta_K^{K^*, \sigma} n_{[\mathbf{x}_L, \mathbf{x}_{K^*}]} \cdot \mathbf{e}_i \bar{u}(\mathbf{x}_K) \\ & + \left( n_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]}^{(L, K^*, \sigma)} + \beta_L^{K^*, \sigma} n_{[\mathbf{x}_L, \mathbf{x}_{K^*}]} \right) \cdot \mathbf{e}_i \bar{u}(\mathbf{x}_L) \end{aligned} \right\} \quad (32)$$

In Equations (30)-(32), the vectors  $n_{[\cdot, \cdot]}$  which are outward normal vectors, their lengths are equal to the corresponding edge lengths. The three coefficients  $\beta_K^{K^*, \sigma}$ ,  $\beta_L^{K^*, \sigma}$  and  $\beta_{K^*}^{K^*, \sigma}$  are defined in Equation (13). We also define  $\sigma = [\mathbf{x}_K, \mathbf{x}_L]$  and the common edge  $K|L$ .

Similarly, we transform the integral of  $\bar{u}$  on  $(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})$  into two integrals on two sub-triangles  $(\mathbf{x}_K, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)$  and  $(\mathbf{x}_L, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)$

$$\int_{(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})} \frac{\partial \bar{u}}{\partial x_i}(\mathbf{x}) d\mathbf{x} = \int_{(\mathbf{x}_K, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \frac{\partial \bar{u}}{\partial x_i}(\mathbf{x}) d\mathbf{x} + \int_{(\mathbf{x}_L, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \frac{\partial \bar{u}}{\partial x_i}(\mathbf{x}) d\mathbf{x}, \quad (33)$$

In the above equation, two integrals of the right hand side are computed by

$$\begin{aligned} \int_{(\mathbf{x}_K, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \frac{\partial \bar{u}}{\partial x_i}(\mathbf{x}) d\mathbf{x} &= \int_{[\mathbf{x}_K, \mathbf{x}_{K^*}]} \bar{u}(\mathbf{e}_i \cdot \bar{\mathbf{n}}_{[\mathbf{x}_K, \mathbf{x}_{K^*}]}) d\gamma(\mathbf{x}) + \int_{[\mathbf{x}_K, \mathbf{x}_\sigma]} \bar{u}(\mathbf{e}_i \cdot \bar{\mathbf{n}}_{[\mathbf{x}_K, \mathbf{x}_\sigma]}) d\gamma(\mathbf{x}), \\ &+ \int_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]} \bar{u}(\mathbf{e}_i \cdot \bar{\mathbf{n}}_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]}^{(K, K^*, \sigma)}) d\gamma(\mathbf{x}). \end{aligned} \quad (34)$$

and

$$\begin{aligned} \int_{(\mathbf{x}_L, \mathbf{x}_{K^*}, \mathbf{x}_\sigma)} \frac{\partial \bar{u}}{\partial x_i}(\mathbf{x}) d\mathbf{x} &= \int_{[\mathbf{x}_L, \mathbf{x}_{K^*}]} \bar{u}(\mathbf{e}_i \cdot \bar{\mathbf{n}}_{[\mathbf{x}_L, \mathbf{x}_{K^*}]}) d\gamma(\mathbf{x}) + \int_{[\mathbf{x}_L, \mathbf{x}_\sigma]} \bar{u}(\mathbf{e}_i \cdot \bar{\mathbf{n}}_{[\mathbf{x}_L, \mathbf{x}_\sigma]}) d\gamma(\mathbf{x}) \\ &+ \int_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]} \bar{u}(\mathbf{e}_i \cdot \bar{\mathbf{n}}_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]}^{(K, K^*, \sigma)}) d\gamma(\mathbf{x}), \end{aligned} \quad (35)$$

where the vectors  $\bar{\mathbf{n}}_{[\cdot, \cdot]}$  are outward normal unit vectors of the considered triangle, and  $\bar{\mathbf{n}}_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]}^{(K, K^*, \sigma)} + \bar{\mathbf{n}}_{[\mathbf{x}_{K^*}, \mathbf{x}_\sigma]}^{(L, K^*, \sigma)} = \mathbf{0}$ . Besides, we have the following relationship between  $\bar{\mathbf{n}}_{[\cdot, \cdot]}$  and  $\mathbf{n}_{[\cdot, \cdot]}$

$$\begin{aligned} n_{[\mathbf{x}_L, \mathbf{x}_\sigma]} &= m([\mathbf{x}_L, \mathbf{x}_\sigma]) \bar{\mathbf{n}}_{[\mathbf{x}_L, \mathbf{x}_\sigma]}, \quad n_{[\mathbf{x}_K, \mathbf{x}_\sigma]} = m([\mathbf{x}_K, \mathbf{x}_\sigma]) \bar{\mathbf{n}}_{[\mathbf{x}_K, \mathbf{x}_\sigma]}, \\ n_{[\mathbf{x}_L, \mathbf{x}_{K^*}]} &= m([\mathbf{x}_L, \mathbf{x}_{K^*}]) \bar{\mathbf{n}}_{[\mathbf{x}_L, \mathbf{x}_{K^*}]}, \quad n_{[\mathbf{x}_K, \mathbf{x}_{K^*}]} = m([\mathbf{x}_K, \mathbf{x}_{K^*}]) \bar{\mathbf{n}}_{[\mathbf{x}_K, \mathbf{x}_{K^*}]} \end{aligned} \quad (36)$$

From Equations (30)-(36), they lead

$$\begin{aligned}
& \int_{(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})} \partial_{D, \mu}^{(i)} \bar{u}_h d\mathbf{x} - \int_{(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})} \frac{\partial \bar{u}}{\partial x_i}(\mathbf{x}) d\mathbf{x} = \\
& = \frac{1}{2} (n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]} \cdot \mathbf{e}_i) \left( \bar{u}(\mathbf{x}_{K^*}) + \bar{u}(\mathbf{x}_K) - \frac{2}{m[\mathbf{x}_{K^*}, \mathbf{x}_K]} \int_{[\mathbf{x}_{K^*}, \mathbf{x}_K]} \bar{u}(\mathbf{x}) d\gamma(\mathbf{x}) \right) \\
& + \frac{1}{2} (n_{[\mathbf{x}_{K^*}, \mathbf{x}_L]} \cdot \mathbf{e}_i) \left( \bar{u}(\mathbf{x}_{K^*}) + \bar{u}(\mathbf{x}_L) - \frac{2}{m[\mathbf{x}_{K^*}, \mathbf{x}_L]} \int_{[\mathbf{x}_{K^*}, \mathbf{x}_L]} \bar{u}(\mathbf{x}) d\gamma(\mathbf{x}) \right) \\
& + \frac{1}{2} (n_{[\mathbf{x}_K, \mathbf{x}_L]} \cdot \mathbf{e}_i) \left( \bar{u}(\mathbf{x}_K) + \bar{u}(\mathbf{x}_L) - \frac{2}{m[\mathbf{x}_K, \mathbf{x}_L]} \int_{[\mathbf{x}_K, \mathbf{x}_L]} \bar{u}(\mathbf{x}) d\gamma(\mathbf{x}) \right) \\
& + \frac{1}{2} (n_{[\mathbf{x}_K, \mathbf{x}_L]} \cdot \mathbf{e}_i) \underbrace{\left[ \bar{u}(\mathbf{x}_K) \left( \frac{m[\mathbf{x}_L, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} - \beta_K^{K^*, \sigma} \right) + \bar{u}(\mathbf{x}_L) \left( \frac{m[\mathbf{x}_K, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} - \beta_L^{K^*, \sigma} \right) - \beta_{K^*}^{K^*, \sigma} \bar{u}(\mathbf{x}_{K^*}) \right]}_{[\bar{u}(\mathbf{x}_K) - \bar{u}(\mathbf{x}_{K^*})] \left( \frac{m[\mathbf{x}_L, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} - \beta_K^{K^*, \sigma} \right) + [\bar{u}(\mathbf{x}_L) - \bar{u}(\mathbf{x}_{K^*})] \left( \frac{m[\mathbf{x}_K, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} - \beta_L^{K^*, \sigma} \right)} \\
& \tag{37}
\end{aligned}$$

because of  $\beta_K^{K^*, \sigma} + \beta_L^{K^*, \sigma} + \beta_{K^*}^{K^*, \sigma} = 1$  and  $\frac{m[\mathbf{x}_K, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} + \frac{m[\mathbf{x}_L, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} = 1$ .

Note that we have the useful results for Equation (37)

$$\lim_{h \rightarrow 0} \left( \frac{m[\mathbf{x}_L, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} - \beta_K^{K^*, \sigma} \right) = \lim_{h \rightarrow 0} \left( \frac{m[\mathbf{x}_K, \mathbf{x}_\sigma]}{m[\mathbf{x}_K, \mathbf{x}_L]} - \beta_L^{K^*, \sigma} \right) = \lim_{h \rightarrow 0} \beta_{K^*}^{K^*, \sigma} = 0 \tag{38}$$

which is proven by Lemma 5.1 in [12], and

$$|\bar{u}(\mathbf{x}_K) - \bar{u}(\mathbf{x}_{K^*})| \leq C_{39} \|\nabla \bar{u}\|_{L^2(K^*)} \quad \text{for all } K \in \mathcal{M}, K^* \in \mathcal{M}^*, \tag{39}$$

it is shown by Theorem 9.12 (Morrey) in [2].

For the other computations of Eq.(37), let us give another triangular element  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_M) \in \mathcal{M}_{K^*}^{**}$ . Two triangles  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_L)$  and  $(\mathbf{x}_{K^*}, \mathbf{x}_K, \mathbf{x}_M)$  have a common edge  $[x_{K^*}, x_K]$ , so we should rewrite the vector  $n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]}$  by  $n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]}^{(K, M, K^*)}$ . This help us distinguish the normal outward vector  $n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]}^{(K, L, K^*)}$  of  $(\mathbf{x}_K, \mathbf{x}_L, \mathbf{x}_{K^*})$  and  $n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]}^{(K, M, K^*)}$  of  $(\mathbf{x}_K, \mathbf{x}_M, \mathbf{x}_{K^*})$ .

Beside, we have the important property of the two normal vectors

$$n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]}^{(K, L, K^*)} + n_{[\mathbf{x}_{K^*}, \mathbf{x}_K]}^{(K, M, K^*)} = 0$$

which also appears in the other edge having a common vertex  $\mathbf{x}_{K^*}$ .

Combining the above property, (29), (37) and (38), we can estimate the right hand size of Equation(28), as follows:

$$\begin{aligned} & \left| \int_{K^*} \partial_{\mathcal{D}, \mu}^{(i)} \bar{u}_h \, d\mathbf{x} - \int_{\partial K^*} \bar{u}(\mathbf{x}) \mathbf{e}_i \cdot n(\mathbf{x}) \, d\mathbf{x} \right| = \\ & \leq \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{K^*}^*} |(n_{\sigma}^{K^*} \cdot \mathbf{e}_i)| \left| \left( \bar{u}(\mathbf{x}_1^{\sigma}) + \bar{u}(\mathbf{x}_2^{\sigma}) - \frac{2}{m(\sigma)} \int_{\sigma} \bar{u}(\mathbf{x}) \, d\mathbf{x} \right) + C_5 h \|\nabla \bar{u}\|_{L^2(K^*)} \right|, \end{aligned} \quad (40)$$

where the points  $\mathbf{x}_1^{\sigma}$ ,  $\mathbf{x}_2^{\sigma}$  are two vertices of an edge  $\sigma$ ,  $m(\sigma)$  is its length, and  $n_{\sigma}^{K^*}$  denotes the outward normal vector of  $K^*$  at  $\sigma$ .

In Equation (40), we remain estimate

$$\psi_{\sigma}(\bar{u}) = \bar{u}(\mathbf{x}_1^{\sigma}) + \bar{u}(\mathbf{x}_2^{\sigma}) - \frac{2}{m(\sigma)} \int_{\sigma} \bar{u}(\mathbf{x}) \, d\gamma(\mathbf{x}) \quad (41)$$

Let  $T$  be the reference triangle with the three vertices  $\mathbf{x}_1^T(0, 0)$ ,  $\mathbf{x}_2^T(1, 0)$ ,  $\mathbf{x}_3^T(0, 1)$ , and we put  $\theta$  be the affine mapping from  $T_{K^*, \sigma} = (\mathbf{x}_1^{\sigma}, \mathbf{x}_2^{\sigma}, \mathbf{x}_{K^*})$  to  $T$  such that  $\theta(\mathbf{x}_1^{\sigma}) = \mathbf{x}_1^T$ ,  $\theta(\mathbf{x}_2^{\sigma}) = \mathbf{x}_2^T$  and  $\theta(\mathbf{x}_{K^*}) = \mathbf{x}_3^T$ .

On the reference triangle  $T$ , the operator  $\hat{\psi}$ , which is defined in  $(H^2(T))'$ , satisfies the operator  $\hat{\psi}(\hat{u}) = \psi_{\sigma}(\hat{u} \circ \theta)$  and

$$\hat{\psi}_{\sigma}(\hat{u}) = \hat{u}(\mathbf{x}_1^T) + \hat{u}(\mathbf{x}_2^T) - \frac{2}{m([\mathbf{x}_1^T, \mathbf{x}_2^T])} \int_{[\mathbf{x}_1^T, \mathbf{x}_2^T]} \hat{u}(\mathbf{x}) \, d\gamma(\mathbf{x}) \quad (42)$$

for all  $\hat{u} \in H^2(T)$ .

Thanks to the Bramble-Hilbert Lemma, we get the following estimation

$$|\hat{\psi}_{\sigma}(\hat{u})| \leq C_6 |\hat{u}|_{H^2(T)}, \quad (43)$$

where there exist the positive constant be independence with anything.  
Using Inequality (43) to estimate  $\psi_\sigma(\bar{u})$ , to this aim, we choose  $\hat{u} = \bar{u} \circ \theta^{-1} \in H^2(T)$ .  
It implies  $\Psi_\sigma(\bar{u}) = \Psi_\sigma(\hat{u})$  and

$$|\Psi_\sigma(\bar{u})| \leq C_7 |\hat{u}|_{H^2(T)}. \quad (44)$$

In order to complete the estimation (44), we use the theorems 3.1.2, 3.1.3 in [5] to give the following results

$$|\hat{u}|_{H^2(T)} \leq \|\theta^{-1}\|^2 \left( \frac{m(T)}{m(T_{K^*,\sigma})} \right)^{1/2} |\bar{u}|_{H^2(T_{K^*,\sigma})}, \quad (45)$$

and  $\|\theta^{-1}\| \leq \frac{\text{diam}(T_{K^*,\sigma})}{\hat{\rho}}$ , where  $\hat{\rho}$  is a diameter of a inscribed circle in  $T$ .  
Applying Inequality (45),  $m(T) = 1$ ,  $\hat{\rho} = 1$  and the geometrical condition (10) to Inequality (44), then it yields

$$|\Psi_\sigma(\bar{u})| \leq C_7 \text{diam}(T_{K^*,\sigma}) |\bar{u}|_{H^2(T_{K^*,\sigma})} \leq C_7 h |\bar{u}|_{H^2(T_{K^*,\sigma})}. \quad (46)$$

From the estimation (46) for each edge  $\sigma \in \mathcal{E}_{K^*}^*$  and the geometrical condition (8), the right hand side of Inequality (40) is less than  $(C_7 h \|u\|_{H^2(K^*)} + C_5 h \|\nabla \bar{u}\|_{L^2(K^*)})$ .  
Moreover, let us any  $K^* \in \mathcal{M}^*$ , and  $\sigma \in \mathcal{E}_{K^*}^*$ ,  $\sigma$  is also an edge of the triangular mesh  $\mathcal{M}^{**}$ , it leads

$$|(n_\sigma^{K^*} \cdot \mathbf{e}_i)| \leq h \quad \text{for all } i = 1, 2 \quad (47)$$

By the inequalities (40), (46) and (47), we have

$$\begin{aligned} \left| \int_{K^*} \text{div}_{\mathcal{D},\mu} \bar{u}_h \, d\mathbf{x} - \int_{\partial K^*} \bar{u}(\mathbf{x}) \mathbf{e}_i \cdot n(\mathbf{x}) \, d\mathbf{x} \right| &\leq \frac{C_7 h^2}{2} \sum_{\sigma \in \mathcal{E}_{K^*}^*} |\bar{u}|_{H^2(T_{K^*,\sigma})} \\ &+ \frac{C_5 h^2}{2} \|\nabla \bar{u}\|_{L^2(K^*)}, \end{aligned} \quad (48)$$

Remark that Inequality (48) only require  $\bar{u} \in H^2(T) \cap H_0^1(\Omega)$  for all triangle  $T \in \mathcal{M}^{**}$  with  $\bigcup_{T \in \mathcal{M}^{**}} \bar{T} = \bar{\Omega}$ .

Together  $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ , it follows

$$\left| \int_{K^*} \text{div}_{\mathcal{D},\mu} \bar{u}_h \, d\mathbf{x} - \int_{\partial K^*} \bar{u}(\mathbf{x}) \mathbf{e}_i \cdot n(\mathbf{x}) \, d\mathbf{x} \right| \leq h^2 C_4 [\|u\|_{H^2(K^*)} + \|\nabla \bar{u}\|_{L^2(K^*)}], \quad (49)$$

where a positive constant  $C_4 = \frac{1}{2} \max\{C_5, C_7\}$  is independent on  $h$ , and  $h$  is small enough.  $\square$

**Lemma 4.2 (Stability of the scheme)**

*Under the geometrical conditions for meshes are satisfied. Then, there exist a positive constant (independent on  $h$ ), such that*

$$\sup_{\substack{\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}} \\ \mathbf{v}_h \neq 0}} \frac{\int_{\Omega} (\operatorname{div}_{\mathcal{D}, \mu} \mathbf{v}_h) q_h \, d\mathbf{x}}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} \geq C_{50} \|q_h\|_{L^2(\Omega)} \quad (50)$$

for all  $q_h \in \mathcal{L}_{\mathcal{D}}$ , in which  $P_1(\mathbf{v}_h) = (P_1(v_h^{(1)}), P_1(v_h^{(2)}))$  is defined the traditional interpolation, constructed on  $\mathcal{M}^{**}$ , the basis Lagrange polynomials having the degree 1 and  $\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}}$ .

*Proof*

For the left hand side of (50), we have

$$\begin{aligned} & \sup_{\substack{\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}} \\ \mathbf{v}_h \neq 0}} \int_{\Omega} \frac{(\operatorname{div}_{\mathcal{D}, \mu} \mathbf{v}_h)}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} \\ &= \sup_{\substack{\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}} \\ \mathbf{v}_h \neq 0}} \left\{ \int_{\Omega} \frac{[\operatorname{div}_{\mathcal{D}, \mu} \mathbf{v}_h - \operatorname{div}(P_1(\mathbf{v}_h))]}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} + \int_{\Omega} \frac{\operatorname{div}(P_1(\mathbf{v}_h))}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} \right\} \\ &\geq \sup_{\substack{\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}} \\ \mathbf{v}_h \neq 0}} \left[ \int_{\Omega} \frac{\operatorname{div}(P_1(\mathbf{v}_h))}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} \right] - \sup_{\substack{\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}} \\ \mathbf{v}_h \neq 0}} \left| \int_{\Omega} \frac{[\operatorname{div}(P_1(\mathbf{v}_h) - \operatorname{div}_{\mathcal{D}, \mu} \mathbf{v}_h)]}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} \right| \end{aligned} \quad (51)$$

where let any  $\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}}$ , the function  $P_1(\mathbf{v}_h)$  is a linear combination constructed by Lagrange polynomials of degree one on  $\mathcal{M}^{**}$  and values of all element of  $\mathbf{v}_h$ .

To estimate  $\int_{\Omega} \frac{\operatorname{div}(P_1(\mathbf{v}_h))}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x}$ , we recall, in the construction of the primal  $\mathcal{M}$  and dual  $\mathcal{M}^*$  meshes, we see that each element of  $\mathcal{M}^*$  containing at most fixed number,  $C_1$  – this is the condition (8), triangles of  $\mathcal{M}$ . From this property, each element of  $\mathcal{M}^*$  forms a disjoint polygonal "macroelement". Moreover, according to Definition 3.2, each  $p_h \in \mathcal{L}_{\mathcal{D}_h}$  is piecewise constant on each "macroelement"  $K^* \in \mathcal{M}^*$ . We then apply to the macroelement technique in [6], [15] and Theorem 3 in [7]. This

leads the stability property is satisfied by using the Fortin's trick [3] for checking the inf-sup condition, i.e,

$$\sup_{\substack{\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}} \\ \mathbf{v}_h \neq 0}} \left[ \int_{\Omega} \frac{\operatorname{div}(P_1(\mathbf{v}_h))}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} \right] \geq \sup_{\substack{\mathbf{w} \in (\mathbb{V}_h(\mathcal{M}^{**}))^2 \\ \mathbf{w} \neq 0}} \left[ \int_{\Omega} \frac{\operatorname{div}(\mathbf{w})}{\|\mathbf{w}\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} \right] \geq \eta \|q_h\|_{L^2(\Omega)}, \quad (52)$$

where the positive constant  $\eta$  is independent on  $h$ ,  $\mathbb{V}_h(\mathcal{M}^{**})$  is the finite element space of the standard finite element method on the triangulation  $\mathcal{M}^{**}$ . Remark that

$$(\mathbb{V}_h(\mathcal{M}^{**}))^2 \subseteq \{P_1(\mathbf{v}_h) | \forall \mathbf{v}_h \in \mathcal{H}_{\mathcal{D}}\}$$

Next, we estimate the following integral

$$\begin{aligned} \int_{\Omega} q_h \frac{[\operatorname{div}(P_1(v_h)) - \operatorname{div}_{D,\mu} v_h]}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} d\mathbf{x} &= \sum_{K^* \in \mathcal{M}^*} \int_{K^*} q_h \sum_{i=1}^2 \frac{[\partial_{D,\mu}^{(i)} v_h^{(i)} - \partial^{(i)} P_1(v_h^{(i)})]}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} d\mathbf{x} \\ &\leq \sum_{K^* \in \mathcal{M}^*} |q_{K^*}| \sum_{i=1}^2 h_{\mathcal{D}^{**}}^2 \frac{\frac{C_7}{2} \sum_{\sigma \in \mathcal{E}_{K^*}^*} |P_1(v_h^{(i)})|_{H^2(T_{K^*,\sigma})} + \frac{C_5}{2} \|\nabla P_1(v_h^{(i)})\|_{L^2(K^*)}}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} \\ (48) \quad &\leq \sqrt{h} \sum_{K^* \in \mathcal{M}^*} \left( \frac{h}{\operatorname{diam}(K^*)} \right)^{3/2} \frac{(\operatorname{diam}(K^*))^{3/2}}{(m(K^*))^{3/4}} \frac{C_5}{\sqrt[4]{m(K^*)}} \int_{K^*} |q_h| d\mathbf{x} \sum_{i=1}^2 \frac{\|\nabla P_1(v_h^{(i)})\|_{L^2(K^*)}}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} \\ &\quad \text{because } P_1(v_h^{(i)}) \text{ is a polynomial of degree 1, this implies } |P_1(v_h^{(i)})|_{H^2(T_{K^*,\sigma})} = 0. \\ &\leq (\sqrt{\zeta_{\mathcal{D}^{**}}} C_2)^3 \sqrt{h} \left( \sum_{K^* \in \mathcal{M}^*} \sqrt{m(K^*)} \right)^{1/2} \left( \sum_{K^* \in \mathcal{M}^*} \|q_h\|_{L^2(K^*)}^2 \sum_{i=1}^2 \frac{\|\nabla P_1(v_h^{(i)})\|_{L^2(K^*)}^2}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}^2} \right)^{1/2} \\ &\leq \underbrace{\xi}_{=(\sqrt{\zeta_{\mathcal{D}^{**}}} C_2)^3} \sqrt{h} \left( \sum_{K^* \in \mathcal{M}^*} \sqrt{m(K^*)} \right)^{1/2} \frac{\|\nabla P_1(\mathbf{v}_h)\|_{(L^2(\Omega))^2}}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} \|q_h\|_{L^2(\Omega)}. \quad (53) \end{aligned}$$

From the two inequalities (51) and (53), we get

$$\sup_{\substack{\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}} \\ \mathbf{v}_h \neq 0}} \int_{\Omega} \frac{(\operatorname{div}_{\mathcal{D},\mu} \mathbf{v}_h)}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h \, d\mathbf{x} \geq \left[ \eta - \xi \sqrt{h} \left( \sum_{K^* \in \mathcal{M}^*} \sqrt{m(K^*)} \right)^{1/2} \right] \|q_h\|_{L^2(\Omega)} \quad (54)$$



We assume that there exist  $h_0$  be small enough, such that  $h \leq h_0$

$$h \left( \sum_{K^* \in \mathcal{M}^*} \sqrt{m(K^*)} \right)^{1/2} < \left( \frac{\eta}{\xi} \right)^2. \quad \square$$

## 5. Convergence of the pFECC scheme

In this section we prove that the pair discrete solution  $(\mathbf{u}_h, p_h) \in \mathcal{H}_{\mathcal{D}_h} \times \mathcal{L}_{\mathcal{D}}$  tend to the weak solutions  $(\mathbf{u}, p)$  of the problem (16), as  $h \rightarrow 0$ .

We firstly state the theorem 5.1 to prove the convergence of the velocity.

**Theorem 5.1 (the convergence of the velocity)** *Under hypotheses  $\mathbf{f} \in (L^2(\Omega))^2$ , (2) and (3), let the positive parameter  $\lambda$  be fixed, then  $\mathbf{u}_h$  converges to  $\mathbf{u}$  in  $(L^2(\Omega))^2$ .*

*Proof.*

We will prove there exists a sub-sequence of  $(\mathbf{u}_h)$ , such that this sub-sequence converges to  $\mathbf{u} \in (H_0^1(\Omega))^2$ , as  $h \rightarrow 0$ . For this purpose, in Eq.(17) and (18), we choose  $\mathbf{v}_h = \mathbf{u}_h \in \mathcal{H}_{\mathcal{D}}$  and  $p_h = q_h \in \mathcal{L}_{\mathcal{D}}$ . The two equations are rewritten by

$$\begin{aligned} \int_{\Omega} \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(1)} \cdot \nabla_{\mathcal{D}, \mu} u_h^{(1)} d\mathbf{x} &+ \int_{\Omega} \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(1)} \cdot \nabla_{\mathcal{D}, \mu} u_h^{(1)} dx \\ &- \int_{\Omega} \operatorname{div}_{\mathcal{D}, \mu}(\mathbf{u}_h) p_h d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot P(\mathbf{u}_h) d\mathbf{x}, \end{aligned} \quad (55)$$

and

$$\int_{\Omega} \operatorname{div}_{\mathcal{D}, \mu}(\mathbf{u}_h) p_h d\mathbf{x} = -\lambda h \int_{\Omega} p_h^2 d\mathbf{x}. \quad (56)$$

On the left hand side of (55), we transform the integral depended the discrete pressure  $p_h$  by (56), then we get

$$\begin{aligned} \int_{\Omega} \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(1)} \cdot \nabla_{\mathcal{D}, \mu} u_h^{(1)} d\mathbf{x} &+ \int_{\Omega} \mu(\mathbf{x}) \nabla_{\mathcal{D}, \mu} u_h^{(1)} \cdot \nabla_{\mathcal{D}, \mu} u_h^{(1)} d\mathbf{x} \\ &+ \lambda h \int_{\Omega} p_h^2 d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot (P(u^{(1)})_h, P(u_h^{(2)})) d\mathbf{x}. \end{aligned} \quad (57)$$

By the condition (2) of the viscosity  $\mu$ , the left hand side (LHS) of Equation (57) is estimated by

$$\text{LHS} \geq \underline{\lambda} \left( \|\nabla_{\mathcal{D}, \mu} u_h^{(1)}\|_{(L^2(\Omega))^2}^2 + \|\nabla_{\mathcal{D}, \mu} u_h^{(2)}\|_{(L^2(\Omega))^2}^2 \right) + \lambda h_{\mathcal{D}^{**}} \|p_h\|_{L^2(\Omega)}^2. \quad (58)$$

And its right hand side (RHS) is bounded by

$$\begin{aligned} \text{RHS} &\stackrel{\leq}{\underset{\text{Young}}{}} \beta \|\mathbf{f}\|_{(L^2(\Omega))^2}^2 + \frac{1}{\beta} \left( \|P(u_h^{(1)})\|_{(L^2(\Omega))^2}^2 + \|P(u_h^{(1)})\|_{(L^2(\Omega))^2}^2 \right) \\ &\leq \beta \|\mathbf{f}\|_{(L^2(\Omega))^2}^2 + \frac{C_8}{\beta} \left( \|\nabla_{\mathcal{D},\mu} u_h^{(1)}\|_{(L^2(\Omega))^2}^2 + \|\nabla_{\mathcal{D},\mu} u_h^{(2)}\|_{(L^2(\Omega))^2}^2 \right) \end{aligned} \quad (59)$$

because of the inequality (30) in [12], where the positive constant  $\beta$  is chosen in (62). From the two inequalities (58) and (59), we have

$$\left( \underline{\lambda} - \frac{C_9}{\beta} \right) \left( \|\nabla_{\mathcal{D},\mu} u_h^{(1)}\|_{(L^2(\Omega))^2}^2 + \|\nabla_{\mathcal{D},\mu} u_h^{(2)}\|_{(L^2(\Omega))^2}^2 \right) + \lambda h \|p_h\|_{L^2(\Omega)}^2 \leq \beta \|\mathbf{f}\|_{(L^2(\Omega))^2}^2 \quad (60)$$

Using additionally Inequality (21) in [12], we can estimate Inequality (60) in the discrete  $H^1$  norm  $\|\cdot\|_{1,\mathcal{D}^{**}}$ , as follows

$$\left( \underline{\lambda} - \frac{C_9}{\beta} \right) C_{10} \left( \|u_h^{(1)}\|_{1,\mathcal{D}^{**}}^2 + \|u_h^{(2)}\|_{1,\mathcal{D}^{**}}^2 \right) + \lambda h \|p_h\|_{L^2(\Omega)}^2 \leq \beta \|\mathbf{f}\|_{(L^2(\Omega))^2}^2, \quad (61)$$

where  $C_9, C_{10}$  are not depend on  $h$ .

Note that the two coefficients  $C_9, C_{10}$  are generated from the two inequalities (21), (30) in [12]. Besides, the coefficient  $\beta$  is chosen by

$$\beta = \frac{2C_9}{\underline{\lambda}} > 0 \quad (62)$$

Hence, Inequality (61) follows

$$\left( \|u_h^{(1)}\|_{1,\mathcal{D}^{**}}^2 + \|u_h^{(2)}\|_{1,\mathcal{D}^{**}}^2 \right) \leq \frac{2\beta}{\underline{\lambda}} \|\mathbf{f}\|_{(L^2(\Omega))^2}^2 \quad (63)$$

and

$$\lambda h \|p_h\|_{L^2(\Omega)}^2 \leq \beta \|\mathbf{f}\|_{(L^2(\Omega))^2}^2 \quad (64)$$

With Inequality (63), we obtain the existence of a subsequence of  $\mathbf{u}_h$  and  $\mathbf{u} \in (H_0^1(\Omega))^2$  such that this subsequence of  $\mathbf{u}_h$  converges to  $\mathbf{u}$  in  $(L^2(\Omega))^2$  as  $h \rightarrow 0$ , which is implied from Lemma 5.7 of [9].

Let  $\phi \in (C_c^\infty(\Omega))^2$  such that  $\text{div}(\phi) = 0$ . We suppose that  $h$  is small enough, so that, for all  $K^{**} \in \mathcal{M}^{**}$ , the intersection of the two sets  $K^{**}$  and  $\text{supp}\{\phi\}$  is nonempty,

then  $\partial K^{**} \cap \partial \Omega = \emptyset$ . Additionally, we will give  $\mathbf{v}_h = \boldsymbol{\phi}_h = (\{\boldsymbol{\phi}(\mathbf{x}_K)\}_{K \in \mathcal{M}}, \{\boldsymbol{\phi}(\mathbf{x}_{K^*})\}_{K^* \in \mathcal{M}^*}) \in \mathcal{H}_{\mathcal{D}}$  in two equations (17) and (18). And using the results of Section 5 in [12], they help us show the convergence of the diffusion operator with the variable viscosity  $\mu(\mathbf{x})$ :

$$\begin{aligned} \lim_{h \rightarrow 0} & \left( \int_{\Omega} \mu(x) \nabla_{\mathcal{D}, \mu} u_h^{(1)} \cdot \nabla_{\mathcal{D}, \mu} \phi_h^{(1)} \, d\mathbf{x} + \int_{\Omega} \mu(x) \nabla_{\mathcal{D}, \mu} u_h^{(2)} \cdot \nabla_{\mathcal{D}, \mu} \phi_h^{(2)} \, d\mathbf{x} \right) \\ &= \int_{\Omega} \mu(x) \nabla \mathbf{u} : \nabla \phi \, d\mathbf{x} \end{aligned} \quad (65)$$

Furthermore, we apply the Holder inequality and a result  $\lim_{h \rightarrow 0} S_i(\phi) = 0$  in Corollary of [12] to show

$$\lim_{h \rightarrow 0} \left( \int_{\Omega} \mathbf{f} \cdot \phi_h \, d\mathbf{x} \right) = \int_{\Omega} \mathbf{f} \cdot \phi \, d\mathbf{x} \quad (66)$$

Next, we need to prove

$$\lim_{h \rightarrow 0} \left( \int_{\Omega} p_h \operatorname{div}_{\mathcal{D}, \mu} \phi_h \, d\mathbf{x} \right) = \lim_{h \rightarrow 0} \left[ \int_{\Omega} p_h \left( \nabla_{\mathcal{D}, \mu} \phi_h^{(1)} \cdot \mathbf{e}_1 + \nabla_{\mathcal{D}, \mu} \phi_h^{(2)} \cdot \mathbf{e}_2 \right) d\mathbf{x} \right] = 0. \quad (67)$$

For this purpose, we put

$$G_h^{(i)} = \int_{\Omega} p_h (\nabla_{\mathcal{D}, \mu} \phi_h^{(i)} \cdot \mathbf{e}_i) \, d\mathbf{x} = \sum_{K^* \in \mathcal{M}^*} \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} m(L^{**}) p_{K^*} \left( \nabla_{\mathcal{D}, \mu} \phi_h^{(1)} \cdot \mathbf{e}_i \right), \quad (68)$$

$$\overline{G}^{(i)} = \int_{\Omega} p_h (\nabla \phi^{(i)} \cdot \mathbf{e}_i) \, d\mathbf{x}, \quad (69)$$

where their relation is expressed by

$$\lim_{h \rightarrow 0} \left( G_h^{(i)} - \overline{G}^{(i)} \right) = 0 \quad \text{for each } i = 1, 2. \quad (70)$$

This result is shown by Lemma 5.1, while its condition (91) is satisfied by (64). Besides, we obtain  $\overline{G}^{(1)} + \overline{G}^{(2)} = 0$  because of  $\operatorname{div} \boldsymbol{\phi} = 0$ , which implies (67) is proven.

In the last requirement for proving this theorem, we also need to indicate  $\operatorname{div}(\mathbf{u}) = 0$

a.e in  $\Omega$ .

Let us  $\varphi \in C_c^\infty(\Omega)$ , and the characteristic function  $\varphi_h \in \mathcal{L}_{\mathcal{D}_h}$  be defined by the value  $\varphi(\mathbf{x}_{K^*})$  in  $K^*$ , for all  $K^* \in \mathcal{M}^*$ . Remark that the sequence  $\varphi_h \rightarrow \varphi$  in  $L^2(\Omega)$ , as  $h \rightarrow 0$ , which is proven by

$$\int_{\Omega} |\varphi_h - \varphi(\mathbf{x})|^2 d\mathbf{x} = \sum_{K^* \in \mathcal{M}^*} \int_{K^*} |\varphi(x_{K^*}) - \varphi(\mathbf{x})|^2 d\mathbf{x} \leq C_{11} \sqrt{h} \, m(\Omega) \|\nabla \varphi\|_{L^2(\Omega)}. \quad (71)$$

To get the inequality (71), we thank to Theorem 9.12 (Morrey) in [2],  $\varphi \in C_c^\infty(\Omega)$  and  $|\mathbf{x} - \mathbf{x}_{K^*}| \leq h$  for all  $\mathbf{x} \in K^*$  ( $\mathbf{x}$  must belong to a triangle  $T_{K^*} \in \mathcal{M}_{K^*}^{**}$ ),  $K^* \in \mathcal{M}^*$ .

In Equation (18), we choose  $q_h = \varphi_h$ . This equation is then rewritten by

$$\int_{\Omega} \operatorname{div}_{\mathcal{D}, \mu}(\mathbf{u}_h) \varphi_h d\mathbf{x} = -\lambda h \int_{\Omega} p_h \varphi_h d\mathbf{x}, \quad (72)$$

whose right and left hand sides are put

$$\overline{H}_h = \lambda h \int_{\Omega} p_h \varphi_h d\mathbf{x} \quad (73)$$

and

$$H_h = \int_{\Omega} \operatorname{div}_{\mathcal{D}, \mu}(\mathbf{u}_h) \varphi_h d\mathbf{x} = \int_{\Omega} (\nabla_{\mathcal{D}, \mu} u_h^{(1)} \cdot \mathbf{e}_1 + \nabla_{\mathcal{D}, \mu} u_h^{(2)} \cdot \mathbf{e}_2) \varphi_h d\mathbf{x} \quad (74)$$

For the right hand side  $\overline{H}_h$ , thanks to (64), it follows

$$|\overline{H}_h| \underset{\text{Holder}}{\leq} \lambda h \|p_h\|_{L^2(\Omega)} \|\varphi_h\|_{L^2(\Omega)} \leq \sqrt{\lambda h} \beta \|f\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}, \quad (75)$$

hence, we get

$$\lim_{h \rightarrow 0} \overline{H}_h = 0 \quad (76)$$

For the left hand side  $H_h$ , for each direction  $x_i$ ,  $i = 1, 2$ , we use the triangular inequality to evaluate

$$\left| \int_{\Omega} (\nabla_{\mathcal{D}, \mu} u_h^{(i)} \cdot \mathbf{e}_i) \varphi_h d\mathbf{x} + \int_{\Omega} u^{(i)}(\mathbf{x}) (\nabla \varphi(\mathbf{x}) \cdot \mathbf{e}_i) \right| \leq H_{1,h} + H_{2,h} + H_{3,h}, \quad (77)$$

where the notations  $H_{j,h}$ , for  $j = 1, 3$ , are defined by

$$\begin{aligned} H_{1,h} &= \left| \int_{\Omega} \left( \nabla_{\mathcal{D},\mu} u_h^{(i)} \cdot \mathbf{e}_i \right) (\varphi_h - \varphi) d\mathbf{x} \right| \\ H_{2,h} &= \left| \int_{\Omega} (\nabla_{\mathcal{D},\mu} u_h^{(i)}(\mathbf{x}) \cdot \mathbf{e}_i) \varphi d\mathbf{x} + \int_{\Omega} P(u_h^{(i)})(\mathbf{x}) (\nabla \varphi \cdot \mathbf{e}_i) d\mathbf{x} \right| \\ H_{3,h} &= \left| \int_{\Omega} u^{(i)}(\mathbf{x}) (\nabla \varphi(\mathbf{x}) \cdot \mathbf{e}_i) d\mathbf{x} - \int_{\Omega} P(u_h^{(i)})(\mathbf{x}) (\nabla \varphi(\mathbf{x}) \cdot \mathbf{e}_i) d\mathbf{x} \right| \end{aligned}$$

Two coefficients  $H_{1,h}$ ,  $H_{3,h}$  are estimated by the Holder inequality

$$H_{1,h} \leq \|\nabla_{\mathcal{D},\mu} u_h^{(i)} \cdot \mathbf{e}_i\|_{L^2(\Omega)} \|\varphi_h - \varphi\|_{L^2(\Omega)} \leq \frac{2\beta}{\underline{\lambda}} \|\varphi_h - \varphi\|_{L^2(\Omega)} \quad (78)$$

and

$$H_{3,h} \leq \|\nabla \varphi \cdot \mathbf{e}_i\|_{L^2(\Omega)} \|u_h^{(i)} - u^{(i)}\|_{L^2(\Omega)}. \quad (79)$$

From the above results, we claim that  $H_{1,h}$ ,  $H_{3,h}$  tend to 0, while the sequence  $\varphi_h \rightarrow \varphi$  and  $u_h^{(i)} \rightarrow u^{(i)}$  in  $L^2(\Omega)$ , as  $h \rightarrow 0$ .

Now, we have to prove  $H_{2,h} \rightarrow 0$ , as  $h \rightarrow 0$ . Before, let us introduce the two following sets

$$\mathcal{M}_{\mu}^{**} = \{K \in \mathcal{M}^{**} | \mu \text{ is not continuous on } K\},$$

if  $K \in \mathcal{M}^{**} \setminus \mathcal{M}_{\mu}^{**}$ , then we assume that  $\mu(\mathbf{x}) = \mu_1$  on a triangle  $K_1 = (\mathbf{x}_K, \mathbf{x}_{K^*}, \mathbf{x}_{\sigma})$ , that  $\mu(\mathbf{x}) = \mu_2$  on a triangle  $K_2 = (\mathbf{x}_L, \mathbf{x}_{K^*}, \mathbf{x}_{\sigma})$ . And

$$\mathcal{M}_{\text{const}}^{**} = \{K \in \mathcal{M}^{**} | \Lambda \text{ is constant on } K\}$$

We rewrite

$$H_{2,h} = \hat{H}_{2,h} + \overline{\overline{H}}_{2,h},$$

where

$$\begin{aligned} \hat{H}_{2,h} &= \int_{\Omega} (\nabla_{\mathcal{D},\mu} u_h^{(i)}(\mathbf{x}) \cdot \mathbf{e}_i) \varphi d\mathbf{x} + \int_{\Omega} P_1(u_h^{(i)})(\mathbf{x}) (\nabla \varphi \cdot \mathbf{e}_i) d\mathbf{x} \\ \overline{\overline{H}}_{2,h} &= \int_{\Omega} [P(u_h^{(i)}) - P_1(u_h^{(i)})](\mathbf{x}) (\nabla \varphi \cdot \mathbf{e}_i) d\mathbf{x}. \end{aligned}$$

We also introduce some notations, as follows:  $\varphi_K$  the average value of  $\varphi$  if  $K \in \mathcal{M}^{**} \cap \mathcal{M}_\mu^{**}$ ,  $\varphi_{K_1}$  (resp.  $\varphi_{K_2}$ ) the average value of  $\varphi$  on  $K_1$  (resp.  $K_2$ ) if  $K^{**} \in \mathcal{M}_\mu^{**}$ ,  $\varphi_{M,N}$  with  $(M, N) \in S_K$ , the average value of  $\varphi$  on  $\vec{\tau}_{M,N,(M,N) \in S_K}$ . We express  $\hat{H}_{2,h}$  by the sum  $\hat{H}_{2,h}^{(1)} + \hat{H}_{2,h}^{(2)} + \hat{H}_{2,h}^{(3)}$  defined by

$$\begin{aligned} \hat{H}_{2,h}^{(1)} &= \sum_{K \in \mathcal{M}_{const}^{**}} |K| \nabla_K u_h^{(i)} \cdot (\varphi_K \mathbf{e}_i) \\ &+ \sum_{K \in \mathcal{M}^{**} \setminus \{\mathcal{M}_{const}^{**} \cup \mathcal{M}_\Lambda^{**}\}} \left[ |K_1| \nabla_{K_1} u_h^{(i)} \cdot (\varphi_{K_1} \mathbf{e}_i) + |K_2| \nabla_{K_2} u_h^{(i)} \cdot (\varphi_{K_2} \mathbf{e}_i) \right] \\ &+ \sum_{K \in \mathcal{M}_\Lambda^{**}} \left[ |K_1| \nabla_{K_1} u_h^{(i)} \cdot (\varphi_{K_1} \mathbf{e}_i) + |K_2| \nabla_{K_2} u_h^{(i)} \cdot (\varphi_{K_2} \mathbf{e}_i) \right] \\ \hat{H}_{2,h}^{(2)} &= \sum_{K \in \mathcal{M}^{**}} \left[ (u_K^{(i)} - u_{K^*}^{(i)}) \vec{\tau}_{K,K^*} \cdot (\varphi_{K,K^*} \mathbf{e}_i) + (u_{K^*}^{(i)} - u_L^{(i)}) \vec{\tau}_{K^*,L} \right. \\ &\quad \left. + (u_L^{(i)} - u_K^{(i)}) \vec{\tau}_{L,K} \cdot (\varphi_{K,L} \mathbf{e}_i) \cdot (\varphi_{K,L} \mathbf{e}_i) \right] \\ \hat{H}_{2,h}^{(3)} &= \sum_{K \in \mathcal{M}^{**}} \left[ \int_{A_{i_K}} [\nabla_{P_1,K} u^{(i)} \cdot (x - x_K)] (\nabla \varphi \cdot \mathbf{e}_i) dx \right. \\ &\quad \int_{A_{i_L}} [\nabla_{P_1,L} u^{(i)} \cdot (x - x_L)] (\nabla \varphi \cdot \mathbf{e}_i) dx \\ &\quad \left. \int_{A_{i_{K^*}}} [\nabla_{P_1,K^*} u^{(i)} \cdot (x - x_{K^*})] (\nabla \varphi \cdot \mathbf{e}_i) dx \right] \end{aligned}$$

Using the computational results of  $T_1$ ,  $T_2$ ,  $T_3$  represented in the pages 27, 28 of [12], we obtain

$$|\hat{H}_{2,h}^{(1)} + \hat{H}_{2,h}^{(2)}| \leq C_{12} \|u\|_{1,\mathcal{D}^{**}} (h_{\mathcal{D}^{**} + \epsilon_2(h)}) \quad (80)$$

with  $\lim_{h \rightarrow 0} \varepsilon_2(h) = 0$ ,

$$\hat{H}_{2,h}^{(3)} \leq h \|\nabla_{P_1} u_h^{(i)}\|_{(L^2(\Omega))^2} \|\nabla \varphi \cdot \mathbf{e}_i\|_{L^2(\Omega)}. \quad (81)$$

With  $\overline{\overline{H}}_{2,h}$ , we also use the results (28) of [12] and (60) to get

$$\overline{\overline{H}}_{2,h} \leq \|P(u_h^{(i)}) - P_1(u_h^{(i)})\|_{L^2(\Omega)} \|\nabla \varphi \cdot \mathbf{e}_i\|_{L^2(\Omega)} \leq C_{13} h \|\mathbf{f}\|_{(L^2(\Omega))^2} \|\nabla \varphi \cdot \mathbf{e}_i\|_{L^2(\Omega)} \quad (82)$$

From (80)-(82), we obtain

$$H_{2,h} \rightarrow 0, \quad \text{as } h \rightarrow 0, \quad (83)$$

together the convergence  $H_{1,h}$ ,  $H_{3,h}$  to 0, we conclude that

$$\int_{\Omega} (\nabla_{\mathcal{D},\mu} u_h^{(1)} \cdot \mathbf{e}_1 + \nabla_{\mathcal{D},\mu} u_h^{(2)} \cdot \mathbf{e}_2) \varphi_h \, d\mathbf{x} \rightarrow - \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) = \int_{\Omega} \operatorname{div}(\mathbf{u})(\mathbf{x}) \varphi(\mathbf{x}), \quad (84)$$

as  $h \rightarrow 0$ .

Therefore, the results (72), (76) and (84) imply

$$\int_{\Omega} \operatorname{div} \mathbf{u}(\mathbf{x}) \varphi(\mathbf{x}) = 0, \quad \text{for all } \varphi \in C^\infty(\Omega). \quad (85)$$

From the results (65), (66), (67) and (85), we proved that the approximate solution  $\mathbf{u}_h$  converges to the weak analysis solution  $\mathbf{u}$ .  $\square$

**Theorem 5.2 (the convergence of the pressure)** *Under hypotheses  $\mathbf{f} \in (L^2(\Omega))^2$ , (2) and (3), let the positive parameter  $\lambda$  be fixed, then, the approximate pressure  $p_h$  converges to  $p$  in  $L^2(\Omega)$ .*

*Proof*

In Equation (17), let any  $\bar{\phi} \in (C^2(\Omega) \cap H_0^1(\Omega))^2$ , we choose

$$\mathbf{v}_h = \bar{\phi}_h = (\{\bar{\phi}(\mathbf{x}_K)\}_{K \in \mathcal{M}}, \bar{\phi}(\mathbf{x}_{K^*})\}_{K^* \in \mathcal{M}^*}),$$

so these equations are rewritten by

$$\int_{\Omega} (\mu(\mathbf{x}) \nabla_{\mathcal{D},\mu}(\mathbf{u}_h)) : \nabla_{\mathcal{D},\mu} \bar{\phi}_h \, d\mathbf{x} - \int_{\Omega} (\operatorname{div}_{\mathcal{D},\mu} \bar{\phi}_h) p_h \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot P(\bar{\phi}_h) \, d\mathbf{x} \quad (86)$$

and in Equation (16),  $\mathbf{v} = \mathbf{P}_1(\bar{\phi}_h)$

$$\int_{\Omega} \mu(\mathbf{x}) \nabla \mathbf{u} : \nabla \mathbf{P}_1(\bar{\phi}_h) \, d\mathbf{x} - \int_{\Omega} \operatorname{div}(\mathbf{P}_1(\bar{\phi}_h)) p \, dx = \int_{\Omega} \mathbf{f} \cdot P_1(\bar{\phi}_h) \, d\mathbf{x}, \quad (87)$$

Equation (86) is subtracted to (27) equals

$$\begin{aligned} & \int_{\Omega} (\mu(\mathbf{x}) \nabla_{\mathcal{D},\mu} \mathbf{u}_h) : \nabla_{\mathcal{D},\mu} \bar{\phi}_h \, d\mathbf{x} - \int_{\Omega} \mu(\mathbf{x}) \nabla \mathbf{u} : \nabla P_1(\bar{\phi}_h) \, d\mathbf{x} \\ & - \int_{\Omega} (\operatorname{div}_{\mathcal{D},\mu} \bar{\phi}_h) p_h \, d\mathbf{x} + \int_{\Omega} \operatorname{div}(P_1(\bar{\phi}_h)) p \, dx = \int_{\Omega} \mathbf{f} \cdot (P(\bar{\phi}_h) - P_1(\bar{\phi}_h)) \, d\mathbf{x}. \end{aligned}$$

This equation corresponds to the following equation

$$O_{1,h} + O_{2,h} + O_{3,h} + O_{4,h} + O_{5,h} = O_{6,h}, \quad (88)$$

where  $P_1(\bar{\phi}_h)$  is different from 0,  $O_{i,h}$ ,  $i = \overline{1,6}$ , are defined, as follows:

$$\begin{aligned} O_{1,h} &= \frac{1}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \int_{\Omega} (\mu(\mathbf{x}) \nabla_{\mathcal{D},\mu} \mathbf{u}_h) : (\nabla_{\mathcal{D},\mu} \bar{\phi}_h - \nabla \bar{\phi}) d\mathbf{x}, \\ O_{2,h} &= \frac{1}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \int_{\Omega} (\mu(\mathbf{x}) \nabla \mathbf{u}) : (\nabla \bar{\phi} - \nabla P_1(\bar{\phi}_h)) d\mathbf{x}, \\ O_{3,h} &= \frac{1}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \int_{\Omega} (\mu(\mathbf{x}) \nabla \bar{\phi}) : (\nabla_{\mathcal{D},\mu} \mathbf{u}_h - \nabla \mathbf{u}) d\mathbf{x}, \\ O_{4,h} &= \frac{1}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \int_{\Omega} (\operatorname{div} P_1(\bar{\phi}_h) - \operatorname{div}_{\mathcal{D},\mu} (\Pi_h \bar{\phi})) p d\mathbf{x}, \\ O_{5,h} &= \frac{1}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \int_{\Omega} \mathbf{f} \cdot [P_1(\bar{\phi}_h) - P(\bar{\phi}_h)] d\mathbf{x}, \\ O_{6,h} &= \frac{1}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \int_{\Omega} (\operatorname{div}_{\mathcal{D},\mu} \bar{\phi}_h) (p_h - p) d\mathbf{x}. \end{aligned}$$

Using the Korn inequality, we get

$$\begin{aligned} |O_{1,h}| &\leq \bar{\lambda} \frac{\|\nabla_{\mathcal{D},\mu} \mathbf{u}_h\|_{(L^2(\Omega))^2}}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \|\nabla_{\mathcal{D},\mu} \bar{\phi}_h - \nabla \bar{\phi}\|_{(L^2(\Omega))^2}, \\ |O_{2,h}| &\leq \bar{\lambda} \frac{\|\nabla \mathbf{u}\|_{(L^2(\Omega))^2}}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \|\nabla \bar{\phi} - \nabla P_1(\bar{\phi}_h)\|_{(L^2(\Omega))^2}, \\ |O_{3,h}| &\leq \bar{\lambda} \frac{\|\nabla_{\mathcal{D},\mu} \mathbf{u}_h\|_{(L^2(\Omega))^2}}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \|\nabla \bar{\phi} - \nabla_{\mathcal{D},\mu} \Pi_h(\bar{\phi})\|_{(L^2(\Omega))^2} \\ &\quad + \frac{1}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \left| \int_{\Omega} (\mu(\mathbf{x}) \nabla_{\mathcal{D},\mu} (\bar{\phi}_h) : \nabla_{\mathcal{D},\mu} \mathbf{u}_h) d\mathbf{x} - \int_{\Omega} (\mu(\mathbf{x}) \nabla \bar{\phi}) : \nabla \mathbf{u} d\mathbf{x} \right|, \\ |O_{4,h}| &\leq \frac{\|p\|_{(L^2(\Omega))^2}}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} (\|\operatorname{div} P_1(\bar{\phi}_h) - \operatorname{div} \bar{\phi}\|_{(L^2(\Omega))^2} + \|\operatorname{div} \bar{\phi} - \operatorname{div}_{\mathcal{D},\mu} (\Pi_h \bar{\phi})\|_{(L^2(\Omega))^2}) \\ |O_{5,h}| &\leq \frac{\|\mathbf{f}\|_{(L^2(\Omega))^2}}{\|P_1(\bar{\phi}_h)\|_{(H^1(\Omega))^2}} \|P_1(\bar{\phi}_h) - P(\bar{\phi}_h)\|_{(L^2(\Omega))^2}. \end{aligned}$$



When  $h_{\mathcal{D}^{**}}$  tends to 0, we have

- $\|\nabla_{\mathcal{D},\mu}\bar{\phi}_h - \nabla\bar{\phi}\|_{(L^2(\Omega))^2} \rightarrow 0$  because of Lemma 4.3 in [9],
- $\|\nabla\bar{\phi} - \nabla P_1(\bar{\phi}_h)\|_{(L^2(\Omega))^2} \rightarrow 0$ ,  $\|\operatorname{div} P_1(\bar{\phi}_h) - \operatorname{div}\bar{\phi}\|_{(L^2(\Omega))^2} \rightarrow 0$

because of Theorems 3.4.3, 3.4.4 in [13].

- $\|\operatorname{div}\bar{\phi} - \operatorname{div}_{\mathcal{D},\mu}(\bar{\phi}_h)\|_{(L^2(\Omega))^2} \rightarrow 0$ ,  $\|P_1(\bar{\phi}_h) - P(\bar{\phi}_h)\|_{(L^2(\Omega))^2} \rightarrow 0$

and  $\|P_1(\bar{\phi}_h)\|_{(L^2(\Omega))^2} \rightarrow \|\bar{\phi}\|_{(L^2(\Omega))^2}$  because of Proposition 5.3, Corollary 5.4 in [12]

- $\left| \int_{\Omega} (\mu(\mathbf{x}) \nabla_{\mathcal{D},\mu}(\bar{\phi}_h) : \nabla_{\mathcal{D},\mu} \mathbf{u}_h) d\mathbf{x} - \int_{\Omega} (\mu(\mathbf{x}) \nabla\bar{\phi} : \nabla \mathbf{u}) d\mathbf{x} \right| \rightarrow 0$

because of the above Lemma 4.1

Moreover,  $\|\nabla_{\mathcal{D},\mu} \mathbf{u}_h\|_{(L^2(\Omega))^2}$  is upper bounded by the positive constant independent  $h_{\mathcal{D}^{**}}$ , which is implied from (60)-(62).

In order to  $O_{6,h}$ , we rewrite Lemma 4.2, as follows: let any  $\epsilon > 0$ , there then exists  $\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}}$ , such that

$$\frac{\int_{\Omega} (\operatorname{div}_{\mathcal{D},\mu} \mathbf{v}_h) q_h d\mathbf{x}}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} + \epsilon \geq C_{14} \|q_h\|_{L^2(\Omega)}$$

With  $\mathbf{v}_h \in \mathcal{H}_{\mathcal{D}}$  satisfied the above inequality, we apply Theorem 1 and Corollary in [16] and Theorem 2 in [17] to generate  $\varphi$  a triangular  $C^2$ -element of degree 9 such that  $\varphi \in (C^2(\Omega) \cap H_0^1(\Omega))^2$  and  $\mathbf{v}_h = \varphi_h = (\{\varphi(\mathbf{x}_K)\}_{K \in \mathcal{M}}, \{\varphi(\mathbf{x}_{K^*})\}_{K^* \in \mathcal{M}^*})$ . This implies that

$$\int_{\Omega} \frac{\operatorname{div}_{\mathcal{D},\mu} \mathbf{v}_h}{\|P_1(\mathbf{v}_h)\|_{(H^1(\Omega))^2}} q_h d\mathbf{x} = \int_{\Omega} \frac{\operatorname{div}_{\mathcal{D},\mu} \varphi_h}{\|P_1(\varphi_h)\|_{(H^1(\Omega))^2}} q_h d\mathbf{x}.$$

Hence,

$$\int_{\Omega} \frac{\operatorname{div}_{\mathcal{D},\mu} \varphi_h}{\|\varphi_h\|_{(H^1(\Omega))^2}} q_h d\mathbf{x} + \epsilon \geq C_{14} \|q_h\|_{L^2(\Omega)},$$

this corresponds to

$$\sup_{\substack{\varphi \in (C^2(\Omega) \cap H_0^1(\Omega))^2 \\ \varphi \neq 0}} \int_{\Omega} \frac{\operatorname{div}_{\mathcal{D},\mu} \varphi_h}{\|P_1(\varphi_h)\|_{(H^1(\Omega))^2}} q_h d\mathbf{x} \geq C_{14} \|q_h\|_{L^2(\Omega)}. \quad (89)$$

From Equations (88), (89), the above estimations of  $|O_{i,h}|$  with  $i = \overline{1, 5}$  and Inequality (50), we obtain

$$\|p_h - p\|_{L^2(\Omega)} \rightarrow 0, \quad \text{as } h \rightarrow 0. \quad \square \quad (90)$$

**Lemma 5.1** *For a given sequence  $p_h \in \mathcal{L}_{\mathcal{D}_h}$ , it satisfies*

$$\|p_h\|_{L^2(\Omega)} \leq \frac{C_{15}}{\sqrt{h}}, \quad (91)$$

where the constant  $C_{15}$  is positive, then

$$\lim_{h \rightarrow 0} \left( \sum_{K^* \in \mathcal{M}^*} \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} m(L^{**}) p_{K^*,h} \left( \nabla_{\mathcal{D},\mu} \phi_h^{(i)} \cdot \mathbf{e}_i \right) - \int_{\Omega} p_h \left( \nabla \phi^{(i)} \cdot \mathbf{e}_i \right) d\mathbf{x} \right) = 0 \quad (92)$$

for each  $i = 1, 2$ , where the above vector function  $\boldsymbol{\phi} = (\phi^{(1)}, \phi^{(2)})$  satisfies the same conditions as  $\boldsymbol{\phi}$  in Theorem 5.1, and the integral  $\int_{\Omega} p_h \left( \nabla_{\mathcal{D},\mu} \phi^{(i)} \cdot \mathbf{e}_i \right) d\mathbf{x}$  is written by

the formula  $\sum_{K^* \in \mathcal{M}^*} \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} m(L^{**}) p_{K^*,h} \left( \nabla_{\mathcal{D},\mu} \phi_h^{(i)} \cdot \mathbf{e}_i \right)$ .

Note that the approximate pressure  $p_h$  satisfies the condition (91), because of (61).

*Proof.*

We will define the two notations  $G_{1,h}^{(i)}$  and  $G_{2,h}^{(i)}$ , as follows:

$$G_{1,h}^{(i)} = \sum_{K^* \in \mathcal{M}^*} \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} m(L^{**}) p_{K^*,h} \left( \nabla_{\mathcal{D},\mu} \phi_h^{(1)} \cdot \mathbf{e}_i \right) \text{ and } G_{2,h}^{(i)} = \int_{\Omega} p_h \left( \nabla \phi^{(i)} \cdot \mathbf{e}_i \right) d\mathbf{x}.$$

We then have their computations

$$\begin{aligned} |G_{1,h}^{(i)} - G_{2,h}^{(i)}| &= \left| \sum_{K^* \in \mathcal{M}^*} p_{K^*,h} \left[ \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} m(L^{**}) \left( \nabla_{\mathcal{D},\mu} \phi_h^{(i)} \cdot \mathbf{e}_i \right) - \int_{K^*} \left( \nabla \phi^{(i)} \cdot \mathbf{e}_i \right) d\mathbf{x} \right] \right| \\ &\leq \sum_{K^* \in \mathcal{M}^*} \sqrt{\mathfrak{m}_{K^*}} |p_{K^*,h}| \frac{1}{\sqrt{\mathfrak{m}_{K^*}}} \left| \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} m(L^{**}) \left( \nabla_{\mathcal{D},\mu} \phi_h^{(i)} \cdot \mathbf{e}_i \right) - \int_{\partial K^*} \phi^{(i)} \left( \bar{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{e}_i \right) d\mathbf{x} \right| \end{aligned} \quad (93)$$

Applying Lemma 4.1 to the term  $G_{K^*}^{(i)}$  in the right hand side of the inequality (93)

$$\begin{aligned} |G_{K^*}^{(i)}| &= \left| \sum_{L^{**} \in \mathcal{M}_{K^*}^{**}} m(L^{**}) \left( \nabla_{\mathcal{D}, \mu} \phi_h^{(i)} \cdot \mathbf{e}_i \right) - \int_{\partial K^*} \phi^{(i)} (\bar{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{e}_i) \, d\mathbf{x} \right| \\ &\leq C_4 h |\phi^{(i)}|_{H^2(K^*)} \end{aligned} \quad (94)$$

and using the following estimation for any  $K^* \in \mathcal{M}^*$

$$\frac{h}{\sqrt{m(K^*)}} \leq \frac{h}{\sqrt{m(L^{**})}} \leq \frac{h}{\text{diam}(L^{**}) \sqrt{C_3}} \leq \frac{\zeta_{\mathcal{D}^{**}}}{\sqrt{C_3}} \quad (95)$$

with any  $L^{**} \in \mathcal{M}_{K^*}^{**}$ , they lead to

$$\begin{aligned} |G_{1,h}^{(i)} - G_{2,h}^{(i)}| &\leq \sum_{K^* \in \mathcal{M}^*} \sqrt{m(K^*)} |p_{K^*}| \frac{h^2}{\sqrt{m(K^*)}} C_4 |\phi^{(i)}|_{H^2(K^*)} \\ &\stackrel{\text{Cauchy Schwarz}}{\leq} \|p_h\|_{L^2(\Omega)} \cdot \frac{h \zeta_{\mathcal{D}^{**}}}{\sqrt{C_3}} C_4 |\phi^{(i)}|_{H^2(\Omega)} \stackrel{(91)}{\leq} C_{16} h |\phi^{(i)}|_{H^2(\Omega)}. \end{aligned} \quad (96)$$

Therefore,

$$\lim_{h \rightarrow 0} (G_{1,h}^{(i)} - G_{2,h}^{(i)}) = 0.$$

□

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